Globus An International Journal of Management & IT
A Refereed Research Journal
Vol 9 / No 1 / Jul-Dec 2017 ISSN: 0975-721X

ON CHARACTERIZATION OF GENERALIZED INFORMATION MEASURES

*Anil Kumar Pasrija
**Dr. Seema

Abstract

The concept of information improvement which involves three probability distributions is well known in literature. Later various authors have paid attention in characterizing it along with its generalized forms. In this section we shall give a characterization of generalized information improvement. The proof is omitted as it follows the lines of Theorem.

Introduction

Consider two probability distributions P = (p_1, p_2, …, p_n) $P = (p_1, p_2, \ldots, p_n) \in \Delta_n$ and $Q = (q_1, q_2, \ldots, q_n) \in \Delta_n$

associated with a discrete random variable X taking finite number of values {x_1, x_2, …, x_n} where

$$\Delta_n = \left\{P = (p_1, p_2, \ldots\ldots, p_n): p_i \geq 0, \sum_{i=1}^{n} p_i = 1\right\}.$$

Consider a generalized measure of information between two distributions P and Q given by

$$I_a^c(P;\ Q) = \left(2^{(a-1)c} - 1\right)^{-1}\left\{\left[\sum_{i=1}^{n} p_i^a q_i^{b-a}\right] - 1\right\}, \ldots\ldots (1)$$

Where a > 0, b > 0, a ≠ b, c ≠ 0 are arbitrary parameters.

The following are the particular cases of the measure (1):

(i)

$$\lim_{c\to 0} I_{a,b}^c(P;\ Q) = I_{a,b}^c(P;\ Q) = \frac{1}{a-b}\log_2\left(\sum_{i=1}^{n} p_i^a q_i^{b-a}\right), \quad a \neq b, a, b > 0 \qquad (2)$$

For b = 1,

$$I_{a,b}^c(P;\ Q) = I_{a,1}^c(P;\ Q) = \frac{1}{a-b}\log_2\left(\sum_{i=1}^{n} p_i^a q_i^{1-a}\right) \quad a \neq 1, a, b > 0 \qquad (3)$$

Which is relative information of order a

For a = 1,

$$I_{a,b}^0(P;\ Q) = I_{1,b}^0(P;\ Q) = \frac{1}{a-b}\log_2\left(\sum_{i=1}^{n} p_i\ q_i^{b-1}\right) \quad b \neq 1, b > 0 \qquad (4)$$

Which is an inaccuracy of order b.

(ii) When c = 1,

$$I_{a,b}^1(P;\ Q) = \left(2^{a-b} - 1\right)^{-1}\left\{\left[\sum_{i=1}^{n} p_i^a q_i^{b-a}\right]\right\}, a, b > 0, a \neq b \ (5)$$

Which is a generalized measure of relative information and inaccuracy.

For b = 1,

* *Research Scholar, Sunrise University, Alwar, Rajasthan*
***Supervisor, Sunrise University, Alwar, Rajasthan*

$$I^1_{a,1}(P;\ Q) = \left(2^{a-b} - 1\right)^{-1}\left\{\left[\sum_{i=1}^{n} p_i^a q_i^{1-a} - 1\right]\right\}, a > 0, a \neq 1 \quad (6)$$

Which is a relative information of degree a

For a = 1,

$$I^1_{1,b}(P;\ Q) = \left(2^{1-b} - 1\right)^{-1}\left\{\left[\sum_{i=1}^{n} p_i^a q_i^{b-1}\right]\right\}, b > 0, b \neq 1 \quad (7)$$

Which is an inaccuracy of degree b

(iii)

$$\lim_{a\to 1} I^0_{a,1}(P;\ Q) = \lim_{a\to 1} I^1_{a,1}(P;\ Q) = \sum_{i=1}^{n} p_i \ \log_2(p_i\ /q_i\ ) \quad (8)$$

Which is a relative information, directed divergence or cross-entropy between the distributions P and Q

(iv)

$$\lim_{b\to 1} I^0_{1,b}(P;\ Q) = \lim_{b\to 1} I^1_{1,b}(P;\ Q) = -\sum_{i=1}^{n} p_i \ \log_2(q_i\ ) \quad (9)$$

Which is an inaccuracy measure between the distribution P and Q

Various authors have paid attention in characterizing these measures of information, for brief review refer to Mathai and Rathie, and Taneja. The measure (1) for b =1, c = $\frac{d-1}{a-1}$ has been characterized by Sharma and Mittal and for a = 1, c = $\frac{d-1}{b-1}$ has been characterized by Sharma and Gupta under mean value considerations.

The aim of this study is to give an axiomatic characterization of the measure (1) under simpler approach. The extensions of (1) to three and more probability distributions is also specified. Whereas, the characterization of measure (1) involving only one probability distribution is given by the author.

Characterization

In this chapter we shall characterize the measure (I) under certain axioms. This characterization is given in the following theorem:

The OREM 1

Let $F : \Delta_n^2 \to \mathfrak{R}\ (reals), n \geqslant 2$ be a function satisfying the following axioms:

(1) F(P; Q) = $k_1 \left[\sum_{i=1}^{n} f(p_i,\ q_i)\right]^c$ + k_2, k_1, k_2, c ≠ 0, where f is a real continuous function defined over $[0, 1]^2$; k_1, k_2 are arbitrary constants and c is an arbitrary parameter.

(2) $F(P * U;\ Q * V) = F(P;\ Q) + F(U;\ V) - \frac{1}{k_2} F(P;\ Q)F(U;\ V),$ for all $P, Q \in \Delta_n, U, V \in \Delta_m,$ *and* $P * U, Q * V \in \Delta_{nm}.$

(3) $F(1, 0; \frac{1}{2}, \frac{1}{2}) = 1.$ *Then* $F(P; Q) = I^c_{a,b}(P; Q).$

Proof. From axioms (I) and (II), we have

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i u_j, q_i v_j)\right]^c = \left(-\frac{k_1}{k_2}\right)^{1/c}\left[\sum_{i=1}^{n} f(p_i q_i)\right]^c \cdot \left[\sum_{i=1}^{n} f(u_j v_j)\right]^c,$$

i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i u_j, q_i v_j) = \left(-\frac{k_1}{k_2}\right)^{1/c}\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_i)\, f(u_j v_j) \quad (10)$$

It is easy to see that the functional equation (10) is equivalent to the following:

$$f(pu, qv) = \left(-\frac{k_1}{k_2}\right)^{\frac{1}{c}} f(p, q) f(u, v) \qquad (11)$$

for all reals p. q, u, v ∈ [0, 1].

Now substituting

$$\left(-\frac{k_1}{k_2}\right)^{1/c} f(p, q) = \phi(p, q). \qquad (12)$$

in (11), we get

$$\phi(pu, qv) = \phi(p, q).\phi(u, v) \qquad (13)$$

The most general (non-trivial) continuous solutions of the functional equation (13) are given by

$$\phi(p, q) = p^\alpha q^\beta, \alpha > 0, \beta \neq 0. \qquad (14)$$

For simplicity write α = a and β = b — a. Then

$$\phi(p, q) = p^a q^{b-a}, a > 0, b \neq a. \qquad (15)$$

Finally, axioms (I) and axiom (11), (12) and (15) together give the required result.

Conclusion

The concept of information improvement which involves three probability distributions is well known in literature. Later various authors have paid attention in characterizing it along with its generalized forms. In this section we shall give a characterization of generalized information improvement. The proof is omitted as it follows the lines of Theorem.

The OREM 2

Let G:$\Delta_n^3 \rightarrow \mathcal{R}$ (reals), n ≥ 2 be a function satisfying the following axioms:

(I)$G(P;\ Q;\ R) = k_1[\sum_{i=1}^{n} g\,(p_i, q_i, r_i)]^c + k_2, k_1, k_2, c \neq 0$

where g is acontinuous function defined over $[0, 1]^3$ k_1, k_2 are arbitrary constants and c is an arbitrary parameter.

(II) $G(P * U;\ Q * V;\ R * W) = G(P;\ Q;\ R) + G(U;\ V;\ W) - \frac{1}{k_2} G(P;\ Q;\ R) G(U;\ V;\ W)$,

for all P, Q, R ∈ Δ_n, U, V, W ∈ Δ_m and P * U, Q * V, R * W ∈ Δ_{nm}.

(III) G(p, 1 —p;p, 1 —p;p,1 —p) = 0,0 <p < 1; G(1,0; 1,0; ½ , ½) = 1; and G(1, 0; ½ , ½ ; ½ , ½)= 0.

Then

$$G(P;\ Q;\ R) = I_a^c(P;\ Q;\ R) = \left(2^{(a-1)c} - 1\right)^{-1} \left\{ \left[\sum_{i=1}^{n} p_i \left(\frac{r_i}{q_i} \right)^{1-a} \right]^c - 1 \right\},$$
$$a > 0, a \neq 1, c \neq 0. \quad (16)$$

The particular cases alongwith their characterizations of themeasure. In the similar way we can characterize a generalized measure involving more than three distributions given by

$$H(P;\ Q;\ R_1, R_2, \ldots, R_N) = \frac{\left(2^{(a-1)c} - 1\right)^{-1}}{\boldsymbol{N}} \left\{ \left[\sum_{i=1}^{n} p_i \left(\frac{\prod_{k=1}^{n} r_{ki}}{q_i^N} \right)^{1-a} \right]^c - 1 \right\},$$
$$a > 0, a \neq 1, c \neq 0. \quad (17)$$

for all P, Q, R_k∈ Δ_n k = 1,2,……N ≥ 1.

When N = 1, (17) reduces to (16).

We shall call the measure (17) as generalized information improvement due to N revisions. The concept of information improvement due to N revisions has already been introduced by the author. Characterizations of some of the particular cases of the measure can be seen in author and Arora, author and Duarte, and Vinocha and Goel.

Alternative approach to define generalized information improvement due to N revisions can also be seen as:

$$I_a^c(P;\ Q;\ R_1, R_2, \ldots, R_N) = \frac{I_a^c(P;\ Q;\ R_1) + I_a^c(P;\ Q;\ R_2) + I_a^c(P;\ Q;\ R_N)}{N}, \quad (18)$$

Where $I_a^c(P;\ Q;\ R)$ is as given in (16).

Now (16) and (18) together give

$$I_a^c(P;\ Q;\ R_1, R_2, \ldots, R_N) = \left(2^{(a-1)c} - 1\right)^{-1} \left\{ \frac{1}{N} \sum_{k=1}^{n} \left[\sum_{i=1}^{n} p_i \left(\frac{r_{ki}}{q_i} \right)^{1-a} \right]^c - 1 \right\},$$
$$a > 0, a \neq 1, c \neq 0. \quad (19)$$

In this case also (19) reduces to (16) for N = 1

It still remains an open question to characterize the measure (19) independently.

References

1. Lemon SC, Roy J, Clark MA, Friedmann PD, Rakowski W., Classification and regression tree analysis in public health: methodological review and comparison with logistic regression, Ann. Behav Med., 2003;26(3):172-181.
2. Buness A, Ruschhaupt M, Kuner R, Tresch A., Classification across gene expression microarray studies, BMC Bioinformatics, 2009, Dec 30;10:453.
3. Chipman, H., George, E., and McCulloch, R. (2008), “Bayesian CART Model Search” (with discussion), Journal of the American Statistical Association, 93, 935-960.
4. Kathiresan S, Melander O, Anevski D, Guiducci C, Burtt NP, Polymorphisms associated with cholesterol and risk of cardiovascular events, Engl J Med., 2008; 358 (12):1240-1249.
5. Kooperberg, C., Ruczinski, I., LeBlanc, M. L., and Hsu, L., “Sequence Analysis using Logic Regression,” Genetic Epidemiology, 21, Supplement 1, 626-631, 2011.